Embedding the n-cube in Lower Dimensions

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Let $m = m(n)$ denote the smallest dimension m such that the vertices of the n-dimensional cube can be embedded into E^m in a way that adjacent vertices have distance at most 1 while any two non-adjacent vertices are at distance more than 1. It is proved that $(1+o(1))n/\log_2 n < m(n) < (2\log_2 3+o(1))n/\log_2 n$ holds.

1. Introduction

Define an adjacency relation on Euclidean n-space E^n by

$$x \text{ adj } y \leftrightarrow 0 < \|x-y\| \leq 1 (x, y \in E^n).$$

Then E^n becomes an infinite graph, and every nonempty subset X of E^n induces a subgraph $\langle X\rangle$. An abstract graph G is said to be *embeddable* in E^n if G is isomorphic to some induced subgraph of E^n. The *sphericity* of G, sph(G), is the smallest integer n such that G is embeddable in E^n.

In most cases it is difficult to determine the sphericity for a given graph. So, certain bounds on the sphericity for many types of graph are sought [1, 2].

In this note we consider the sphericity of the n-cube Q_n. Here, the n-cube is a graph isomorphic to the subgraph of E^n induced by the point set $V=\{(s_1, s_2, \ldots, s_n)\colon s_i = 0 \text{ or } 1\}$. ($\langle V\rangle$ itself will be called the *standard n-cube*.)

It is easily seen that $\text{sph}(Q_2)=2$ and $\text{sph}(Q_3)=3$. So one might expect that $\text{sph}(Q_n)=n$ for all n. However, it is shown that $\text{sph}(Q_n)\leq n-1$ for $n\geq 6$. Our main result is that for any $c > 2\log_2 3 = 3{\cdot}1699\ldots$,

$$n/\log_2 n < \text{sph}(Q_n) < cn/\log_2 n$$

as $n\to\infty$.

2 A Few Simple Results

Theorem 1. Sph$(Q_2)=2$, sph$(Q_3)=3$.

Proof. We show only $\text{sph}(Q_3)=3$. Since $\text{sph}(Q_3)\leq 3$ is clear, suppose that Q_3 is realized as an induced subgraph of E^2 whose edges are straight line segments. Then no two edges of Q_3 cross each other, for otherwise, some three of the four endpoints form a triangle contrary to that Q_3 is triangle-free. Similarly, it is seen that the interior of any quadrilateral of Q_3 contains no point of Q_3. Hence the interiors of the six quadrilaterals of Q_3 are mutually disjoint. This implies that the surface of a sphere is topologically embeddable into E^2, which is clearly false. Thus $\text{sph}(Q_3)$ must be greater than 2.

Maybe $\text{sph}(Q_n)=n$ for $n=4, 5$, but we have no proof.

Theorem 2. *Sph$(Q_{n+1})\leq n$ for $n\geq 5$.*

Proof. We will embed Q_{n+1} in E^n for $n\geq 5$. Take Q_n in E^n as the standard n-cube, that is, as the subgraph induced by the set

$$V=\{(s_1, s_2, \ldots, s_n);\ s_i = 0 \text{ or } 1\}.$$

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This will be half the points of Q_{n+1}, and to get the other half, translate it by 1 in the direction of one of the main diagonals. E.g. translate by $\vec{v}=(1/\sqrt{n},\ldots,1/\sqrt{n})$. If $s=(s_1,\ldots,s_n)$ and $t=(t_1,\ldots,t_n)$ are two different points of V, then $|s_i-t_i|=1$ for some i. Since $(1\pm 1/\sqrt{n})^2>1/n$ for $n\geq 5$, it follows that

$$\|s-(t+\vec{v})\|=\sum_{i=1}^{n}(s_i-t_i-1/\sqrt{n})^2>\sum_{i=1}^{n}1/n=1.$$

Thus the union of V and $V+\vec{v}$ induces a graph isomorphic to Q_{n+1}.

REMARK. The above construction leads to an embedding of the whole rectangular lattice, too. Thus the lattice subgraph induced by all integral points of E^{n+1} is also embeddable in E^n for $n\geq 5$.

3. A Lower Bound

THEOREM 3. *For sufficiently large n,* $\mathrm{sph}(Q_n)>n/\log_2 n$.

PROOF. Let v_0 be a point of Q_n. Then within distance $\lfloor n/2\rfloor$ from v_0 there are

$$\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n}{i}\geq 2^{n-1}$$

points, and half of these points are independent since Q_n is a bipartite graph. Hence, if $\mathrm{sph}(Q_n)=m$, then in E^m, at least 2^{n-2} balls of radius $\frac{1}{2}$ must be packed in a ball of radius $\lfloor n/2\rfloor+\frac{1}{2}$. Considering the volumes of these balls, we have

$$2^{n-2}(1/2)^m<[(n+1)/2]^m\Delta_m,$$

where Δ_m is the packing density, that is, the fraction of E^m filled by optimally packed m-balls of equal radius. It is known [4, p. 81] that $\Delta_m<m/(2^{m/2}e)$ for $m\to\infty$, so we have

$$2^{n-2}<(n+1)^m m/(2^{m/2}e).$$

Taking logarithm to the base 2, yields $m>n/\log_2 n$ $(n\to\infty)$.

4. The Upper Bound

THEOREM 4. *Let c be a constant, $c>2\log_2 3=3{\cdot}1699\ldots$. Then* $\mathrm{sph}(Q_n)<cn/\log_2 n$ *as $n\to\infty$.*

PROOF. Set $m=\lfloor cn/\log_2 n\rfloor$ and ω_{ij} $(1\leq i\leq n, 1\leq j\leq m)$ be mn independent random variables, each taking the values $\pm 1/\sqrt{m}$ with probability $\frac{1}{2}$. Then $\omega_i=(\omega_{il},\ldots,\omega_{im})$ $(1\leq i\leq n)$ are independent random unit vectors in E^m. Let $\langle V\rangle$be the standard n-cube in E^n and define a map $\psi: V\to E^m$ by $\psi(s)=s_1\omega_1+\ldots+s_n\omega_n$ $(s=(s_1,\ldots,s_n)\in V)$. We are going to prove that:

$$\mathrm{Prob}(\langle\psi(V)\rangle \text{ is isomorphic to } \langle V\rangle)\to 1, \qquad n\to\infty.$$

For each integer k, $2\leq k\leq n$, let P_k be the probability that $\|\omega_1+\omega_2+\ldots+\omega_k\|\leq 1$. We assume the following lemma, which will be proved later.

LEMMA. $\displaystyle\sum_{k=2}^{n}\binom{n}{k}2^k P_k\to 0, \qquad n\to\infty.$

Now, for $s, t \in V$ let $E_{s,t}$ denote the event '$\|\psi(s)-\psi(t)\| \leq 1$ and let $J = J(s,t) = \{i: s_i \neq t_i\}$. Thus $\# J \geq 2$ is equivalent to $\|s-t\| > 1$. If s, t are adjacent (i.e. $\|s-t\|=1$), then clearly $\|\psi(s)-\psi(t)\|=1$. Hence, to prove the theorem, it is enough to show that

$$\operatorname{Prob}\left(\bigcup_{\# J \geq 2} E_{s,t}\right) \to 0, \qquad n \to \infty,$$

in words, the probability that $E_{s,t}$ happens for some nonadjacent pair $s, t \in V$, tends to zero as $n \to \infty$. Since $\psi(s)-\psi(t) = \sum_{i \in J} \sigma_i \omega_i$ ($\sigma_i = 1$ or -1), if we vary s, t with keeping $J = J(s,t)$ fixed, then $2^{\# J}/2$ different kinds of events would be obtained as $E_{s,t}$. And since all $\pm\omega_i$ are identically distributed, it follows easily that $\operatorname{Prob}(E_{s,t})$ equals P_j, where $j = \# J$. Hence

$$\operatorname{Prob}\left(\bigcup_{\# J = k} E_{s,t}\right) \leq \binom{n}{k} 2^k P_k$$

and

$$\operatorname{Prob}\left(\bigcup_{\# J \geq 2} E_{s,t}\right) \geq \sum_{k=2}^{n} \binom{n}{k} 2^k P_k \to 0, \qquad n \to \infty.$$

5. Proof of the Lemma

The proof will be carried out by splitting the sum $\sum_{k=2}^{n} \binom{n}{k} 2^k P_k$ into two parts.

Case 1 $(2 \leq k \leq m^{1/4})$.
Since

$$\left\|\sum_{i=1}^{k} \omega_i\right\|^2 = k + \sum_{1 \leq h \neq i \leq k} \omega_h \cdot \omega_i,$$

$\|\sum_{i=1}^{k} \omega_i\| \leq 1$ implies that

$$\sum_{1 \leq h \neq i \leq k} \omega_h \cdot \omega_i \leq 1-k.$$

hence

$$\begin{aligned} P_k &\leq \operatorname{Prob}\left(\sum_{1 \leq h \neq i \leq k} \omega_h \cdot \omega_i \leq 1-k\right) \\ &\leq \operatorname{Prob}\left(\omega_h \cdot \omega_i \leq \frac{1-k}{k(k-1)} \text{ for some } h, i\right) \\ &\leq k(k-1) \operatorname{Prob}(\omega_1 \cdot \omega_2 \leq -1/k) \\ &\leq k(k-1) \operatorname{Prob}(|\omega_1 \cdot \omega_2| \geq m^{-1/4}). \end{aligned}$$

Here, $\omega_1 \cdot \omega_2$ is the sum of the m independent random variables $\zeta_j := \omega_{1j}\omega_{2j}$ $(1 \leq j \leq m)$, each takes the values $\pm 1/m$ with probability $\frac{1}{2}$. Then we can apply Bernstein's inequality (see, e.g. [3]).

Bernstein's inequality. Let ζ_j $(1 \leq j \leq m)$ be independent variables satisfying $|\zeta_j - E(\zeta_j)| \leq K$ and let $\zeta = \sum \zeta_j$. Then

$$\operatorname{Prob}(|\zeta - E(\zeta)| \geq \mu D(\zeta)) < 2\exp(-\mu^2/5)$$

holds for every $\mu \leq D(\zeta)/K$. ($E(\zeta)$, $D(\zeta)$ denote, as usual the expectation and the standard deviation of ζ, respectively.)

In our case, $K=1/m$, $E(\zeta)=0$, and $D(\zeta)=1/\sqrt{m}$. Hence letting $\mu=m^{1/4}$, we have

$$\text{Prob}(|\omega_1\cdot\omega_2|\geq m^{-1/4})<2\exp(-\tfrac{1}{5}m^{1/2}).$$

Thus

$$P_k<2k(k-1)\exp(-\tfrac{1}{5}m^{1/2}).$$

Therefore

$$\sum_{k=2}^{\lfloor m^{1/4}\rfloor}\binom{n}{k}2^kP_k<(2n)^{m^{1/4}}m\exp(-\tfrac{1}{5}m^{1/2})$$
$$=m\exp(m^{1/4}(\log_e 2n-\tfrac{1}{5}m^{1/4})).$$

This tends to zero for $n\to\infty$.

Case 2$(m^{1/4}<k\leq n)$.

In this case $k\to\infty$ as $n\to\infty$. Set $f=\log_2 n/m$. Then it is easily verified that $mf\to\infty$ but $mf/k=(1/k)^{1-o(1)}$ as $n\to\infty$. Now let ξ_j be the jth coordinate of $\omega_1+\cdots+\omega_k$. Then

$$\left\|\sum_{i=1}^{k}\omega_i\right\|^2=\sum_{j=1}^{m}\xi_j^2\geq f\cdot\#\{j:|\xi_j|\geq\sqrt{f}\}.$$

Hence $\|\omega_1+\cdots+\omega_k\|\leq 1$ implies that $\#\{j:|\xi_j|\geq\sqrt{f}\}\leq 1/f$, or equivalently, $\#\{j:|\xi_j|<\sqrt{f}\}>m-1/f$.

We first evaluate the probability $p=\text{Prob}(|\xi_j|<\sqrt{f})$. Let ν_j be the number of positive ω_{ij}s $(1\leq i\leq k)$. Then ν_j is distributed according to the binomial distribution with parameters k, $\frac{1}{2}$ and

$$\xi_j=\sum_{i=1}^{k}\omega_{ij}=\frac{1}{\sqrt{m}}(2\nu_j-k).$$

Hence

$$p=\text{Prob}(|\xi_j|<\sqrt{f})=\text{Prob}(|\nu_j-k/2|\nu_j-k/2|<\sqrt{mf}/2)$$
$$<\sqrt{mf}\binom{k}{\lfloor k/2\rfloor}\left(\frac{1}{2}\right)^k\sim\frac{\sqrt{mf}}{\sqrt{\pi k/2}}2^k\left(\frac{1}{2}\right)^k$$
$$<\sqrt{mf/k}=\left(\frac{1}{\sqrt{k}}\right)^{1-o(1)}.$$

Now, if $S\subset\{1,2,\ldots,m\}$ and $\#S>m-1/f$ then

$$\text{Prob}(\{j:|\xi_j|<\sqrt{f}\}=S)=p^{\#S}(1-p)^{m-\#S}<p^{\#S}$$
$$<\left(\frac{mf}{k}\right)^{(m-1/f)/2}=\left(\frac{1}{k}\right)^{(1-o(1))m/2}.$$

Thus

$$P_k<\text{Prob}(\#\{j:|\xi_j|<\sqrt{f}\}>m-1/f)<2^mk^{-(1-o(1))m/2}.$$

Therefore

$$\sum_{k>m^{1/4}}\binom{n}{k}2^kP_k<\sum_{k>m^{1/4}}\binom{n}{k}2^k2^mk^{-(1-o(1))m/2}.$$

Here we split the sum for $k\geq m/10$ $(\sum_1)$ and $k<m/10$ $(\sum_2)$.

$$\textstyle\sum_1<\sum\binom{k}{k}2^k2^m\left(\frac{10}{m}\right)^{(1-o(1))m/2}<2^m3^n\left(\frac{10}{m}\right)^{(1-o(1))m/2}.$$

Since $m \log_2 m = cn(1+o(1))$, $m^m = 2^{cn(1+o(1))}$. And since $2^{c/2} > 3$, we have

$$\sum_1 < \left(\frac{3 \cdot (20)^{c/\log_2 n}}{2^{c(1+o(1))/2}}\right)^n \to 0, \qquad n \to \infty.$$

Finally

$$\sum_2 < \sum_{k<m/10} \binom{n}{k} 2^k 2^m k^{-(1-o(1))m/2}$$
$$< \left(\frac{m}{10}\right) 2^m (2n)^{m/10} m^{-(1-o(1))m/8}.$$

Since $n^{8/9} < m$ for $n \to \infty$, we have $m^{-(1-o(1))m/8} < n^{-m/9}$, and

$$\sum_2 < \left(\frac{m}{10}\right) 2^{2m} n^{-m/90} \to 0, \qquad n \to \infty.$$

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